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The next issue of MATHEMATICS MAGAZINE will contain, in part, the first installment of a semi-expository paper on "The Generalized Weirstrass Approximation Theorem" by Marshall H. Stone, a paper on "Congruence Methods as Applied to Diaphantive Analysis" by H. S. Vandiver, and the first of a series of articles telling in simple language "the story" of various mathematical courses designed to bridge the chasm between the general reader and the abstract world of mathematics.

The Operational Calculus

by F. D. MURNAGHAN

It is now generally recognized that the operational calculus furnishes a powerful method for obtaining particular solutions of a system of linear differential equations with constant coefficients. Most expositions of operational calculus, however, do not make it quite clear just what particular solutions of the system of linear differential equations are furnished by the operational calculus and many are restricted to the case of a single differential equation. We think, then, that an expository article on the operational calculus may serve a useful purpose. The principal novelty in this exposition lies in the fact that we do not confine ourselves to the case of a *normal* system of linear differential equations, i.e., one in which the determinant of the coefficients of the highest order derivative which appears in the system is different from zero. We treat also the *abnormal* case in which this determinant is zero. In the normal case the derivatives of order lower than the highest which occurs in the system may be assigned arbitrary initial values while in the abnormal case these lower order derivatives must satisfy certain relations.

1. *The Laplace Transformation.* Operational calculus is based on an operation (known as the Laplace Transformation) which sends complex valued functions of a *real variable* t into functions of a complex variable z. The functions of the real variable t are defined only for $t \geqslant 0$, i.e., for non-negative values of t, but it is convenient to extend their range of definition by the convention that they are zero when $t<0$. It is sufficient for our purpose to confine our attention to *piecewise-continuous* functions. If $f(t)$ is one of these piecewise-continuous functions we denote by $f(+0)$ the limit at $t=0$ of $f(t)$, it being understood that t tends to zero through *positive* values; thus if $f(t)$ is continuous at $t=0$, $f(+0)=f(0)=0$, but if $f(t)$ is not continuous at $t=0$, $f(+0)$ is not necessarily equal to $f(0)$. We suppose that there exists a complex number $z_0=x_0+y_0i$ such that the improper integral

$$\int_0^\infty e^{-z_0t}f(t)dt$$

exists. Then if $z=x+yi$ is any complex number which lies to the right of z_0 (i.e., for which $x>x_0$) the improper integral

$$\phi(z)=\int_0^\infty e^{-zt}f(t)dt$$

exists and is analytic at z.* We term this integral the Laplace Transform of $f(t)$ and we denote it by $L(f)$. Thus

If $L(f)$ is defined at $z=z_0$, $L(f)$ is analytic over the half-plane $R(z)>R(z_0)$ where $R(z)$ denotes the real part of z.

It follows from the very definition that $L(f)$ is a linear operator, i.e.

$$L(cf)=cL(f)\,;\ c \text{ any constant};$$

$$L(f_1+f_2)=L(f_1)+L(f_2).$$

If we denote by 1 the *unit-function*, i.e., the function which is 1 when, $t>0$ and 0 when $t<0$ (the value at $t=0$ being unimportant) we have

$$L(1)=\frac{1}{z}\,;\quad R(z)>0$$

and if c is any complex number we have

$$L(e^{-ct})=\frac{1}{z+c}\,;\quad R(z+c)>0.$$

It follows readily that

$$L(\cos\alpha t)=\frac{z}{z^2+\alpha^2}\,;\qquad L(\sin\alpha t)=\frac{\alpha}{z^2+\alpha^2}\,;\quad R(z)>|I(\alpha)|$$

where $I(\alpha)$ denotes the imaginary part of the (complex) number α. Furthermore if p is any positive integer

$$L\{t^pf(t)\}=(-1)^p\delta^pL(f)$$

where δ denotes differentiation with respect to z. For example

$$L(t\sin\alpha t)=\frac{2\alpha z}{(z^2+\alpha^2)^2}\,;\quad R(z)>|I(\alpha)|.$$

Of particular importance is the following result:

If $L(f)$ is a proper rational fraction this proper rational fraction has near $z=\infty$, the power series development

$$L(f)=\frac{f(+0)}{z}+\frac{Df(+0)}{z^2}+\cdots+\frac{D^{n-1}f(+0)}{z^n}+\cdots$$

where D denotes differentiation with respect to t and $Df(+0)$, for example, is the limit at $t=0$ of Df, it being understood that $t\to 0$ through positive values.

*The proofs of the statements made in Sections 1, 2 and 5 of this article may be found in the chapter entitled Operational Calculus of the book *Applied Mathematics* by the author, which is now in press.

On the other hand a rational fraction which is not proper is not the Laplace Transform of any function of t (since $|L(f)|$ may be made arbitrarily small by merely making $R(z)$ sufficiently large).

Finally we state the following *uniqueness theorem*:

$f(t)$ is determined unambiguously by its Laplace transform $L(f)$ at any point t at which $f(t)$ is continuous.

This fundamental result is a consequence of the fact that $L(f)$ is, when regarded as a function of y, the Fourier transform of the function of t which is zero when $t<0$ and which $=(2\pi)^{\frac{1}{2}}e^{-xt}f(t)$ when $t>0$..

2. *The convolution process.*

If f and g are any two piecewise-continuous functions of t the integral

$$\int_0^t f(\tau)g(t-\tau)d\tau$$

is a continuous function of t which we denote by $f * g$. It is easy to show that $f * g = g * f$ and we term $f * g$ the *convolution* of f and g. If Df exists and is piecewise-continuous then, at any point t where g is continuous, $f * g$ is differentiable, its derivative being furnished by the formula

$$D(f * g) = f(+0)g + (Df * g).$$

The convolution process is associative, i.e.,

$$(f * g) * h = f * (g * h)$$

and so we denote the common value of these two expressions by the symbol $f * g * h$. The significance of the convolution process for operational calculus lies in the following theorem:

If $L(f)$ and $L(g)$ are both defined at z_0 the Laplace Transform of $f * g$ exists and has the value $L(f)\, L(g)$ at any point z for which $R(z-z_0)>0$.

Since $\quad f * 1 = \displaystyle\int_0^t f(\tau)d\tau$ and since $L(1) = \dfrac{1}{z}, \quad R(z)>0,$

we have
$$L\left\{\int_0^t f(\tau)d\tau\right\} = \frac{L(f)}{z}$$

it being understood that the real part of z is positive and that z lies to the right of a point z_0 at which $L(f)$ is defined. This result may be conveniently phrased as follows:

Integration over $[0,t]$ *is reflected in the domain of Laplace Transforms by division by* z.

It readily follows that if $f(t)$ possesses a piecewise-continuous derivative $D(f)$ and if $D(f)$ possesses a Laplace Transform at $z=z_0$ then at any point z for which $R(z-z_0)>0$, $L(f)$ exists and

$$L(Df)=z\{L(f)\}-f(+0).$$

In particular if $f(+0)=0$ we have $L(Df)=zL(f)$; in other words

Differentiation with respect to t *is reflected (for those functions for which* $f(+0)=0$*) in the domain of Laplace Transforms by multiplication by* z. *If* $f(+0)\neq 0$ *this multiplication by* z *must be followed by the adjustment involved in subtracting* $f(+0)$ *from* $z\,L(f)$.

We conclude with the following extension of this result:

Let $f=f(t)$ possess a piecewise-continuous nth order derivative which possesses a Laplace Transform at $z=z_0$; then $L(f)$ exists at any point z for which $R(z-z_0)>0$ and

$$L(D^n f)=z^n L(f)-z^{n-1}f(+0)-z^{n-2}Df(+0)-\cdots-D^{n-1}f(+0).$$

The particular case $n=2$ is of especial importance in applications of the theory; for this case we have

$$L(D^2 f)=z^2L(f)-z\,f(+0)-Df(+0).$$

3. *The operational solution of a single linear differential equation with constant coefficients.* We write the differential equation, of order n, as follows:

$$P(D)y=f.$$

Here D denotes differentiation with respect to the independent variable t and $P(D)$ is the *polynomial operator.*

$$P(D)=c_0D^n+c_1D^{n-1}+\cdots+c_n.$$

where $c_0,\cdots,c_n$ are constants. $f=f(t)$ is a given piecewise-continuous function of t which is zero if $t<0$ and which possesses at some point $z_0=x_0+y_0i$ a Laplace Transform. The problem is to determine $y=y(t)$ so as to satisfy the differential equation and to possess, together with its derivatives up to the $(n-1)$st, inclusive, prescribed initial values $y(+0)$, $Dy(+0),\cdots, D^{n-1}y(+0)$. To solve this problem we assume that y has a piecewise-continuous nth order derivative D^ny which possesses at some point z, whose real part is greater than x_0, a Laplace Transform. We have, then, the series of relations

$$L(Dy) = z\,L(y) - y(+0)$$

$$L(D^2y) = z^2L(y) - zy(+0) - Dy(+0)$$

$$\cdots$$

$$L(D^ny) = z^nL(y) - z^{n-1}y(+0) - \cdots - D^{n-1}y(+0)$$

and these imply that

$$L\{P(D)y\} = P(z)L(y) - (c_0z^{n-1} + \cdots + c_{n-1})y(+0) - \cdots - c_0D^{n-1}y(+0).$$

Choosing z so that it lies to the right of every zero of the polynomial $P(z)$ we are sure that $P(z) \neq 0$ and so

$$L(y) = \frac{L(f)}{P(z)} + \frac{c_0z^{n-1} + \cdots + c_{n-1}}{P(z)}\,y(+0) + \cdots + \frac{c_0}{P(z)}\,D^{n-1}y(+0)$$

(since $P(D)y = f$). The initial conditions expressed by the statement that y and all its derivatives up to the $(n-1)$st, inclusive, are zero at $t = +0$ yield a particularly simple formula for $L(y)$, namely,

$$L(y) = \frac{L(f)}{P(z)}\ .$$

We shall term the solution of the problem which corresponds to these particularly simple initial conditions the *rest solution* and we shall denote the rest solution by $r = r(t)$. Thus

$$L(r) = \frac{L(f)}{P(z)}\ .$$

If $s_0(t), s_1(t), \cdots, s_{n-1}(t)$ are the functions of t whose Laplace Transforms

are $$\frac{c_0z^{n-1} + \cdots + c_{n-1}}{P(z)}, \cdots, \quad \frac{c_0}{P(z)}$$

respectively, we have

$$L(y) = L(r) + y(+0)L(s_0) + \cdots + D^{n-1}y(+0)L(s_{n-1}).$$

Since any function of t which possesses a Laplace Transform is unambiguously determined, at any point where it is continuous, by this Laplace Transform, it follows that

$$y = r + y(+0)s_0 + \cdots + D^{n-1}y(+0)s_{n-1}.$$

Thus $y = y(t)$ is determined as soon as we have found the *rest solution* $r = r(t)$ and the *auxiliary functions* $(s_0, \cdots, s_{n-1})$. To facilitate the determination of r and of these auxiliary functions we consider the problem

of determining the solution $q=q(t)$ of the associated *homogeneous* differential equation

$$P(D)u=0$$

which is zero together with its derivatives up to the $(n-2)$nd, inclusive, at $t=+0$, while

$$D^{n-1}q(+0)=\frac{1}{c_0}.$$

The same argument that furnished $L(y)$ yields

$$P(z)L(q)=c_0D^{n-1}q(+0)=1$$

so that
$$L(q)=\frac{1}{P(z)}$$

Since $L(r)=\dfrac{L(f)}{P(z)}=L(q)L(f)$ we have

$$r=q*f$$

(for the convolution of two functions is reflected in the domain of Laplace Transforms by simple multiplication). Furthermore since $q(+0)=0$ we have

$$L(Dq)=z\,L(q)=\frac{z}{P(z)}$$

and, similarly,

$$L(D^2q)=\frac{z^2}{P(z)},\cdots,L(D^{n-1}q)=\frac{z^{n-1}}{P(z)}.$$

Hence
$$s_0=c_0D^{n-1}q+\cdots+c_{n-1}q\ ;$$

$$s_1=c_0D^{n-2}q+\cdots+c_{n-2}q\ ;$$

$$\vdots$$

$$s_{n-1}=c_0q.$$

Thus the solution of the problem is furnished, in terms of the function $q=q(t)$, by the formula

$$y=r+(c_0D^{n-1}q+\cdots+c_{n-1})y(+0)+\cdots+c_0qD^{n-1}y(+0)$$

where $r=q*f$.

The function $q = q(t)$ is easily obtained by analysing

$$\frac{1}{P(z)}$$

into simple fractions. In fact the function which has

$$\frac{1}{z-\alpha}$$

as its Laplace Transform is $e^{\alpha t}$; the function which has

$$\frac{1}{(z-\alpha)^p}$$

as its Laplace Transform is

$$\frac{t^{p-1}}{(p-1)!}\, e^{\alpha t};$$

the function which has $\quad \dfrac{1}{(z-a)^2+b^2}$

as its Laplace Transform is $\quad \dfrac{1}{b}\, e^{at} \sin bt$

and the function which has $\quad \dfrac{z}{(z-a)^2+b^2}$

as its Laplace Transform is $\{\cos bt + (a/b)\sin bt\} e^{at}$ and so on. In particular if all the zeros $(\alpha_1, \cdots, \alpha_n)$ of $P(z)$ are *simple* we have

$$P(z) = c_0(z-\alpha_1)\cdots(z-\alpha_n)$$

and

$$\frac{1}{P(z)} = \sum_{k=1}^{n} \frac{1}{\delta P(\alpha_k)} \frac{1}{(z-\alpha_k)}$$

so that

$$q(t) = \sum_{k=1}^{n} \frac{e^{\alpha_k t}}{\delta P(\alpha_k)}\,.$$

We shall denote by $R(t)$ the rest solution obtained when $f(t)$ is the *unit function* 1. Since

$$L(1) = \frac{1}{z}\,.$$

we have
$$L(R)=\frac{1}{zP(z)}\ .$$

Hence
$$R(t)=q * 1=\int_0^t q(\tau)d\tau.$$

However the expression for $R(t)$ may best be found directly without first determining $q(t)$. All we have to do is to analyse

$$\frac{1}{zP(z)}$$

into simple fractions. In the particular case where $P(z)$ has n simple zeros $(\alpha_1, \cdots, \alpha_n)$, *none of which is zero*, we have

$$\frac{1}{zP(z)}=\frac{1}{P(0)z}+\sum_{k=1}^{n}\frac{1}{\alpha_k\delta P(\alpha_k)(z-\alpha_k)}$$

and so
$$R(t)=\frac{1}{P(0)}+\sum_{k=1}^{n}\frac{e^{\alpha_k t}}{\alpha_k\delta P(\alpha_k)}\ .$$

This result is known as the *Heaviside Expansion Formula*; pay particular attention to the fact that the solution it furnishes is the *rest solution*, i.e., the solution which is zero together with all its derivatives up to the $(n-1)$st, inclusive, at $t=+0$.

Since
$$L(r)=\frac{L(f)}{P(z)}=zL(f)\ \frac{1}{z\ P(z)}=z\ L(f)\ L(R)$$

and since $z\ L(f)=L(Df)+f(+0)$ we have
$$L(r)=L(Df)\ L(R)+f(+0)\ L(R).$$

Hence, since convolution is reflected in the domain of Laplace Transforms by simple multiplication,

$$r=f(+0)R+(Df * R)$$

i.e.,
$$r(t)=f(+0)R(t)+\int_0^t Df(\tau)R(t-\tau)d\tau.$$

This important result is known as the Maxwell-Boltzmann-Hopkinson *Principle of Superposition.* It furnishes a convenient method of determining the rest solution $r(t)$ corresponding to a given ***applied impulse*** $f(t)$ when the rest solution $R(t)$ corresponding to the unit applied impulse is known.

Example. $(D^2+n^2)y=A\cos\omega t.$

Here $P(z)=z^2+n^2$. If $n\neq 0$, $q(t)=(1/n)\sin nt$ while, if $n=0$, $q(t)=t$.

Case 1. $n\neq 0$.

Since $c_0=1$, $c_1=0$, $c_2=n^2$ we have

$$s_0=c_0Dq+c_1=\cos nt\ ;\quad s_1=c_0q=(1/n)\sin nt.$$

The determination of $r(t)$ depends on whether $\omega^2=n^2$ or not.

Case 1^a. $\omega^2\neq n^2$.

$$L(r)=\frac{Az}{(z^2+n^2)(z^2+\omega^2)}=\frac{A}{n^2-\omega^2}\left\{\frac{z}{z^2+\omega^2}-\frac{z}{z^2+n^2}\right\}$$

so that
$$r=\frac{A}{n^2-\omega^2}\{\cos\omega t-\cos nt\}.$$

Case 1^b. $\omega^2=n^2$

$$L(r)=\frac{Az}{(z^2+n^2)^2}=-\frac{A}{2}\,\delta\left(\frac{1}{z^2+n^2}\right).$$

Hence
$$r=\frac{A}{2n}\,t\sin nt.$$

Thus the solution in Case 1 is as follows:

Case 1^a. $$y(t)=\frac{A}{n^2-\omega^2}(\cos\omega t-\cos nt)+y(+0)\cos nt+\frac{Dy(+0)}{n}\sin nt\ ;$$

Case 1^b. $$y(t)=\frac{A}{2n}\,t\sin nt+y(+0)\cos nt+\frac{Dy(+0)}{n}\sin nt.$$

Case 2. $n=0$.

Since $q(t)=t$ we have $s_0=1$, $s_1=t$.

Case 2^a. $\omega\neq 0$

$$L(r)=\frac{A}{z(z^2+\omega^2)}=\frac{A}{\omega^2}\left\{\frac{1}{z}-\frac{z}{z^2+\omega^2}\right\}$$

so that
$$r=\frac{A}{\omega^2}(1-\cos\omega t).$$

Case 2^b. $\omega=0$

$$L(r)=\frac{A}{z^3} \text{ so that } r=\frac{At^2}{2}.$$

Thus the solution in Case 2 is as follows:

Case 2^a. $y(t)=\dfrac{A}{\omega^2}(1-\cos\omega t)+y(+0)+Dy(+0)t$;

Case 2^b. $y(t)=\dfrac{At^2}{2}+y(+0)+Dy(+0)t.$

4. *The operational solution of a system of linear differential equations with constant coefficients.* We shall regard a system of m differential equations for m unknowns as a *vector* differential equation (the m unknowns $y^1, y^2, \cdots, y^m$ being the coordinates of the unknown vector $y=y(t)$). The vector differential equation may be written in the form

$$P(D)y=f$$

where, now $P(D)$ is an mxm matrix whose elements $P_r^s(D)$ ($P_r^s(D)$) being the element in the sth row and rth column) are given polynomial differential operators. If the highest order derivative that appears in any of the m^2 elements $P_r^s(D)$ of the matrix $P(D)$ is the nth we may write

$$P(D)=C_0D^n+C_1D^{n-1}+\cdots+C_n$$

where the coefficients $(C_0,C_1\cdots,C_n)$ are, now, constant mxm matrices. C_0 is not the zero matrix but it may very well be singular, i.e., its determinant may be zero. We suppose that the given vector function $f(t)$ possesses at some point z_0 a Laplace Transform and that our vector differential equation possesses a solution $y=y(t)$ having a piecewise-continuous nth order derivative which possesses, if the real part of z is sufficiently large, a Laplace Transform. (We understand by the Laplace Transform of a vector, or of a matrix, the vector, or matrix, whose coordinates are the Laplace Transforms of the corresponding coordinates of the given vector, or matrix.) Proceeding in exactly the same way as in the case of a single differential equation, we obtain the series of relations:

$$L(Dy)=zL(y)-y(+0)$$
$$L(D^2y)=z^2L(y)-z\,y(+0)-Dy(+0)\ ;$$
$$\cdots$$
$$L(D^ny)=z^n(Ly)-z^{n-1}y(+0)-\cdots-D^{n-1}y(+0)$$

and these imply that

$$L\{P(D)y\} = P(z)\,L(y) - (C_0 z^{n-1} + \cdots + C_{n-1})y(+0) - \cdots - C_0 D^{n-1}y(+0).$$

This relation reduces to

$$L\{P(D)y\} = P(z)L(y)$$

if $y(+0)$, $Dy(+0), \cdots, D^{n-1}y(+0)$ satisfy the following relations:

$$C_0 y(+0) = 0\ ;$$

$$C_0 Dy(+0) + C_1 y(+0) = 0\ ;$$

$$C_0 D^{n-1}y(+0) + \cdots + C_{n-1}y(+0) = 0.$$

If C_0 is non-singular these relations yield, one after the other, the relations

$$y(+0) = 0; \quad Dy(+0) = 0; \ \cdots ; D^{n-1}y(+0) = 0.$$

Whether or not C_0 is singular we shall term any solution of our vector differential equation whose initial values satisfy the relations:

$$C_0 y(+0) = 0; \cdots ; C_0 D^{n-1}y(+0) + \cdots + C_{n-1}y(+0) = 0$$

a *rest solution* and we shall denote a rest solution by $r = r(t)$. Then

$$P(z)L(r) = L\{P(D)r\} = L(f).$$

Choosing z so that it lies to the right of every zero of the polynomial *det* $P(z)$ we are sure that $P(z)$ is non-singular. Denoting its reciprocal by $N(z)$ we have

$$L(r) = N(z)L(f).$$

We shall say that our vector differential equation is ***normal*** when C_0 is non-singular and ***abnormal*** when C_0 is singular (so that *det* $C_0 = 0$). When the vector differential equation is normal *det* $P(z)$ is a polynomial in z of degree nm and each cofactor of $P(z)$ is a polynomial in z of degree not greater than $n(m-1)$. Hence each element of $N(z)$ is a ***proper*** rational fraction. When the vector differential equation is abnormal this may not be the case. We shall consider only the case where

Each element of $\dfrac{1}{z}\,N(z)$ *is a proper rational fraction.*

In order to get some idea as to how restrictive this assumption is, consider for a moment the case $m=2$, $n=2$ which occurs, for example, in the study of coupled electric circuits possessing inductance, resistance and capacity. In this case each element of

$$\frac{1}{z} N(z)$$

is a proper rational fraction unless $det\ P(z)$ is a degree less than 2. Since the number of arbitrary constants involved in the solution of the vector differential equation is precisely the degree of $det\ P(z)$, it follows that if any element of

$$\frac{1}{z} N(z)$$

is not a proper rational fraction the number of arbitrary constants involved in the solution is at least three less than the number (4) that would be involved if the vector differential equation were normal.

Since, then, each element of

$$\frac{1}{z} N(z)$$

is, by hypothesis, a proper rational fraction, there exists an mxm matrix $R=R(t)$ whose Laplace Transform is

$$\frac{1}{z} N(z)$$

and we have

$$\frac{1}{z}N(z)=\frac{R(+0)}{z}+\frac{DR(+0)}{z^2}+\cdots+\frac{D^{k-1}R(+0)}{z^k}+\cdots$$

or, equivalently,

$$N(z)=R(+0)+\frac{DR(+0)}{z}+\frac{D^2R(+0)}{z^2}+\cdots.$$

On multiplying (on the left) this relation by $P(z)=C_0z^n+\cdots+C_n$, we obtain the series of relations

$$C_0R(+0)=0;$$

$$C_0DR(+0)+C_1R(+0)=0;$$

$$\cdots$$

$$C_0D^nR(+0)+\cdots+C_nR(+0)=E_m;$$

$$C_0D^{n+1}R(+0)+\cdots+C_nDR(+0)=0$$

$$\cdots$$

In other words:

Each column vector of the mxm matrix $R=R(t)$ satisfies at $t=+0$ the initial conditions that must be satisfied by a rest solution of the vector differential equation.

It follows that

$$L\{P(D)R\}=P(z)L(R)=P(z)\frac{1}{z}N(z)=\frac{E_m}{z}=L(E_m)$$

so that
$$P(D)R=E_m.$$

In other words

$R=R(t)$ is the rest solution of the matric differential equation $P(D)R=E_m$.

If $r=r(t)$ is a rest solution of the given vector differential equation $P(D)y=f$ we have

$$L(r)=N(z)\ L(f)=L(R)\ z\ L(f)=L(R)\{LD(f)+f(+0)\}$$

and so
$$r=Rf(+0)+(R*Df)$$

where we understand by $(R*Df)$ the following

$$(R*Df)=\int_0^t R(\tau)Df(t-\tau)d\tau.$$

This is the ***Principle of Superposition*** (of rest solutions) for a vector differential equation. That $r=r(t)$ defined by the formula

$$r=Rf(+0)+(R*Df)$$

actually is a rest solution of the vector differential equation

$$P(D)y=f$$

is an immediate consequence of the facts that $P(D)R=E_m$ and that each column vector of R satisfies the initial conditions appropriate to a rest solution. (We assume that f possesses a piecewise-continuous nth order derivative when evaluating $D^n r$.)

If we multiply the relation

$$N(z)=R(+0)+\frac{DR(+0)}{z}+\frac{D^2R(+0)}{z^2}+\cdots$$

on the *right* by $P(z)=C_0z^n+\cdots+C_n$ we obtain the series of relations

$$R(+0)C_0=0;$$

$$DR(+0)C_0+R(+0)C_1=0;$$

$$\cdots$$

$$D^nR(+0)C_0+\cdots+R(+0)C_n=E_m;$$

$$D^{n+1}R(+0)C_0+\cdots+DR(+0)C_n=0.$$

On multiplying, then, the equation $P(D)y=f$ on the left by $R(+0)$ it follows from the first of the relations above that

$$R(+0)\{C_1D^{n-1}+\cdots+C_n\}y=R(+0)f.$$

Hence the initial values of $y, Dy,\cdots, D^{n-1}y$ must satisfy the relation

$$R(+0)C_1D^{n-1}y(+0)+\cdots+R(+0)C_ny(+0)=R(+0)f(+0).$$

If the equation is normal (so that $det\ C_0\neq 0$) $R(+0)$ is the zero matrix and this relation is vacuous. If $n>1$ we obtain an additional relation connecting the initial values of $y, Dy,\cdots, D^{n-1}y$ by multiplying the equation $P(D)y=f$ on the left by $DR(+0)$ and adding to this the relation

$$R(+0)\{C_1D^n+\cdots+C_nD\}y=R(+0)Df$$

(obtained by differentiating with respect to t the relation just given) and then using the second of the relations given above. If $n>2$, another such relation is obtained by multiplying the equation $P(D)y=f$ on the left by $D^2R(+0)$ and eliminating D^ny by using the first three of the relations given above and so on. We suppose in what follows that the initial values of $y, Dy,\cdots, D^{n-1}y$ satisfy, when the given vector differential equation is abnormal, these necessary relations.

We now introduce the *mxm* matrix $Q=DR$ each of whose elements is the derivative with respect to t of the corresponding element of R. Since

$$L\{P(D)Q\}=L\{D\ P(D)R\}=z\ L\{P\ D(R)\}-\{P\ D(R)\}_{t=0}$$

$$=E_m-E_m=0$$

we have $P(D)Q=0$. In other words:

Each column vector of the mxm matrix Q is a solution of the associated homogeneous vector differential equation $P(D)u=0$.

It follows that each column vector of the matrices $DQ, D^2Q,\cdots$ is a solution of this homogeneous equation Hence each column vector of the matrix

$$S_0=D^{n-1}QC_0+D^{n-2}QC_1+\cdots+QC_{n-1}$$

$$=D^nRC_0+\cdots+DRC_{n-1}$$

is a solution of the associated homogeneous vector differential equation Furthermore

$$S_0(+0) = D^nR(+0)C_0 + \cdots + DR(+0)C_{n-1} = E_m - R(+0)C_n$$

and $DS_0(+0) = -DR(+0)C_n$; $D^2S(+0) = -D^2R(+0)C_n$ and so on.

Similarly the mxm matrix

$$S_1 = D^{n-2}QC_0 + \cdots + QC_{n-2} = D^{n-1}RC_0 + \cdots + DRC_{n-2}$$

is such that each of its column vectors is a solution of the associated homogeneous differential equation and

$$S_1(+0) = -R(+0)C_{n-1}; \quad DS_1(+0) = E_m - R(+0)C_n - DR(+0)C_{n-1},$$

$D^2S_1(+0) = -DR(+0)C_n - D^2R(+0)C_{n-1}$ and so on. Continuing in this way we construct the matrices $S_3, \cdots$, ending with $S_{n-1} = QC_0$ and we set up the vector

$$y = r + S_0y(+0) + S_1\, Dy(+0) + \cdots + S_{n-1}D^{n-1}y(+0)$$

where r is the rest solution of the equation $P(D)y = f$. Since each column vector of the various matrices $S_0, \cdots, S_{n-1}$ is a solution of the homogeneous equation $P(D)u = 0$, y is a solution of the non-homogeneous equation $P(D)y = f$. Since $r(+0) = R(+0)\, f(+0)$ (by virtue of the Principle of Superposition) the value of y at $t = +0$ is

$$R(+0)f(+0) + \{E_m - R(+0)C_n\}y(+0)$$
$$-R(+0)C_{n-1}Dy(+0) - \cdots - R(+0)C_1D^{n-1}y(+0) = y(+0).$$

(for $R(+0)\{C_1D^{n-1}y(+0) + \cdots + C_ny(+0)\} = R(+0)f(+0)$) Continuing this argument we see that the vector

$$y = r + S_0y(+0) + \cdots + S_{n-1}D^{n-1}y(+0)$$

is the solution of the vector differential equation $P(D)y = f$ which assumes, together with its derivatives up to the $(n-1)$st inclusive, at $t = +0$ the (properly assigned) values $y(+0)$, $Dy(+0), \cdots D^{n-1}y(+0)$.

Note. When the vector differential equation is normal the initial values of R and of its derivatives up to the $(n-1)$st, inclusive, are zero while $D^nR(+0) = C_0^{-1}$. It follows that

$$S_0(+0) = E_m, \quad DS_0(+0) = 0, \cdots, D^{n-1}S_0(+0) = 0;$$

Similarly $S_1(+0) = 0, \quad DS_1(+0) = E_m, \cdots, D^{n-1}S_1(+0) = 0;$

$$\cdots\cdots$$

$$S_{n-1}(+0) = 0, \cdots, D^{n-1}S_{n-1}(+0) = E_m.$$

Since, when the vector differential equation is normal, $R(+0)=0$ each element of $N(z)$ is a proper rational fraction and

$$L(Q)=L(DR)=zL(R)-R(+0)=zL(R)=N(z).$$

In this case, then, it is easier to obtain Q directly by analyzing the elements of $N(z)$ into simple fractions and then R follows from the relation

$$R(t)=\int_0^t Q(\tau)d\tau.$$

Since $L(r)=N(z)\ L(f)$ the Principle of Superposition may be put, for a normal vector differential equation, in the form

$$r=Q*f.$$

Example 1.

$$(D-1)x-2y=t$$

$$-2x+(D-1)y=t$$

Here $n=1$ and $C_0=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$. Thus the vector differential equation is normal and $x(+0)$, $y(+0)$ may be assigned arbitrarily. Let us set $x(+0)=2$, $y(+0)=4$. We have

$$P(z)=\begin{pmatrix}z-1 & -2\\ -2 & z-1\end{pmatrix},\ det\ P(z)=z^2-2z-3=(z-3)(z+1)$$

so that $$N(z)=P^{-1}(z)=\begin{pmatrix}\dfrac{z-1}{(z-3)(z+1)} & \dfrac{2}{(z-3)(z+1)}\\[2ex] \dfrac{2}{(z-3)(z+1)} & \dfrac{z-1}{(z-3)(z+1)}\end{pmatrix}$$

$$=\begin{pmatrix}\dfrac{1}{2(z-3)}+\dfrac{1}{2(z+1)} & \dfrac{1}{2(z-3)}-\dfrac{1}{2(z+1)}\\[2ex] \dfrac{1}{2(z-3)}-\dfrac{1}{2(z+1)} & \dfrac{1}{2(z-3)}+\dfrac{1}{2(z+1)}\end{pmatrix}.$$

Since $L(Q)=N(z)$ we have

$$Q(t)=\begin{pmatrix}\frac{1}{2}e^{3t}+\frac{1}{2}e^{-t} & \frac{1}{2}e^{3t}-\frac{1}{2}e^{-t}\\ \frac{1}{2}e^{3t}-\frac{1}{2}e^{-t} & \frac{1}{2}e^{3t}+\frac{1}{2}e^{-t}\end{pmatrix}$$

Since $f=v(t,t)$ (by which we mean that f is the vector whose coordinates are t, t) we have $Q(t-\tau)f(\tau)=v(e^{3(t-\tau)}\tau,\ e^{3(t-\tau)}\tau)$ so that

$$r=Q*f=\int_0^t Q(t-\tau)f(\tau)d\tau$$

$$=v\big((1/9)e^{3t}-(1/3)t-1/9,\quad (1/9)e^{3t}-(1/3)t-1/9\big).$$

Since $C_0=E_2$ the desired solution is obtained by adding to r the result of operating on $v(2,4)$ by $Q(t)$, namely, $v(3e^{3t}-e^{-t},\ 3e^{3t}+e^{-t})$. Hence

$$x=(28/9)e^{3t}-e^{-t}-(1/3)t-1/9;\quad y=(28/9)e^{3t}+e^{-t}-(1/3)t-1/9.$$

Example 2.

$$(2D-1)x+(3D-2)y=te^t$$

$$(2D+1)x+(3D+2)y=te^{2t}$$

This is an abnormal first order vector differential equation since the matrix $C_0=\begin{pmatrix}2 & 3\\ 2 & 3\end{pmatrix}$ is singular. Since $P(z)=\begin{pmatrix}2z-1 & 3z-2\\ 2z+1 & 3z+2\end{pmatrix}$ we have *det* $P(z)=2z$ and so

$$N(z)=P^{-1}(z)=\begin{pmatrix}\dfrac{3}{2}+\dfrac{1}{z} & -\dfrac{3}{2}+\dfrac{1}{z}\\[2ex] -1-\dfrac{1}{2z} & 1-\dfrac{1}{2z}\end{pmatrix}.$$

$$\text{Hence}\qquad L(R)=\frac{N(z)}{z}=\begin{pmatrix}\dfrac{3}{2z}+\dfrac{1}{z^2} & -\dfrac{3}{2z}+\dfrac{1}{z^2}\\[2ex] -\dfrac{1}{z}-\dfrac{1}{2z^2} & \dfrac{1}{z}-\dfrac{1}{2z^2}\end{pmatrix}$$

so that

$$R(t)=\begin{pmatrix}\dfrac{3}{2}+t & -\dfrac{3}{2}+t\\[2ex] -1-\dfrac{1}{2}t & 1-\dfrac{1}{2}t\end{pmatrix}$$

Hence $Q(t)=\begin{pmatrix}1 & 1\\ -\dfrac{1}{2} & -\dfrac{1}{2}\end{pmatrix}$. Since $R(+0)=\begin{pmatrix}\dfrac{3}{2} & -\dfrac{3}{2}\\ -1 & 1\end{pmatrix}$,

$C_1=\begin{pmatrix}-1 & -2\\ 1 & 2\end{pmatrix}$ we have $R(+0)C_1=\begin{pmatrix}-3 & -6\\ 2 & 4\end{pmatrix}$ so that the initial values $x(+0)$, $y(+0)$ of x and y must satisfy the relation $-3x(+0)-6y(+0)=0$ or, equivalently, $x(+0)+2y(+0)=0$. Since $Q(t-\tau)f(\tau)=v\left(\tau(e^\tau+e^{2\tau}),\ -\frac{1}{2}\tau\ (e^\tau+e^{2\tau})\right)$ we have

$$Q*f=v\left(\frac{5}{4}+e^t(t-1)+\frac{1}{4}e^{2t}(2t-1),\ -\frac{5}{8}-\frac{1}{2}e^t(t-1)-\frac{1}{8}e^{2t}\ (2t-1)\right).$$

Hence $r=R(+0)f+Q*f$

$$=v\left(\frac{5}{4}+\frac{1}{2}e^t(5t-2)-\frac{1}{4}e^{2t}(4t+1),\ -\frac{5}{8}-\frac{1}{2}e^t(3t-1)+\frac{1}{8}e^{2t}(6t+1)\right).$$

Since $Q(t)C_0=\begin{pmatrix}4 & 6\\ -2 & -3\end{pmatrix}$ the general solution is obtained by adding $v\left(4x(+0)+6y(+0),\ -2x(+0)-3y(+0)\right)$ to $r(t)$. Taking into account the relation connecting $x(+0)$ and $y(+0)$ we find

$$x=\frac{5}{4}+\frac{1}{2}e^t(5t-2)-\frac{1}{4}e^{2t}(4t+1)+x(+0);$$

$$y=-\frac{5}{8}-\frac{1}{2}e^t(3t-1)+\frac{1}{8}e^{2t}(6t+1)+y(+0).$$

Note. In this example $f(+0)=0$ so that, even though $R(+0)\neq 0$, $r(+0)=0$ since $r(+0)=R(+0)\ f(+0)$. Hence y is found by adding $v\,(x(+0),\ y(+0)$ to r.

Example 3.

$$(2D^2-D+9)x-(D^2+D+3)y=0$$

$$(2D^2+D+7)x-(D^2-D+5)y=0.$$

Here $C_0=\begin{pmatrix}2 & -1\\ 2 & -1\end{pmatrix}$ so that the second order vector differential equation is abnormal. $P(z)=\begin{pmatrix}2z^2-z+9 & -(z^2+z+3)\\ 2z^2+z+7 & -(z^2-z+5)\end{pmatrix}$,

det $P(z)=6z^3-6z^2+24z-24=6(z-1)(z^2+4)$. Hence

$$N(z)=P^{-1}(z)=\frac{1}{6}\begin{pmatrix}\dfrac{-(z^2-z+5)}{(z-1)(z^2+4)} & \dfrac{z^2+z+3}{(z-1)(z^2+4)}\\[2ex] \dfrac{-(2z^2+z+7)}{(z-1)(z^2+4)} & \dfrac{2z^2-z+9}{(z-1)(z^2+4)}\end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} \frac{1}{z^2+4} - \frac{1}{z-1} & \frac{1}{z^2+4} + \frac{1}{z-1} \\ -\frac{1}{z^2+4} - \frac{2}{z-1} & -\frac{1}{z^2+4} + \frac{2}{z-1} \end{pmatrix}$$

Since each element of $N(z)$ is a proper rational fraction we have $R(+0) = 0$,

$$\left(\text{since } N(z) = R(+0) + \frac{DR(+0)}{z} + \cdots\right), \text{ and so } L(Q) = N(z).$$

Hence $$Q(t) = \frac{1}{6}\begin{pmatrix} \frac{1}{2}\sin 2t - e^t & \frac{1}{2}\sin 2t + e^t \\ -\frac{1}{2}\sin 2t - 2e^t & -\frac{1}{2}\sin 2t + 2e^t \end{pmatrix}$$

and so

$$R(t) = \frac{1}{6}\begin{pmatrix} -\frac{1}{4}(\cos 2t - 1) - (e^t - 1) & -\frac{1}{4}(\cos 2t - 1) + e^t - 1 \\ \frac{1}{4}(\cos 2t - 1) - 2(e^t - 1) & \frac{1}{2}(\cos 2t - 1) + 2(e^t - 1) \end{pmatrix}.$$

Since

$$D\,R(+0) = Q(+0) = \begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}, \quad C_1 = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 9 & -3 \\ 7 & -5 \end{pmatrix},$$

we have $D\,R(+0)\,C_1 = \begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix}$, $D\,R(+0)\,C_2 = \begin{pmatrix} -2 & -2 \\ -4 & -4 \end{pmatrix}$. Hence $x(+0)$, $y(+0)$, $Dx(+0)$, $Dy(+0)$ are connected by the relation

$$-2x(+0) - 2y(+0) + 2Dx(+0) + 2Dy(+0) = 0$$

or, equivalently,

$$x(+0) + y(+0) - Dx(+0) - Dy(+0) = 0.$$

Since f is the zero vector, $r(t)$ is the zero vector. Again

$$QC_0 = \frac{1}{6}\begin{pmatrix} 2\sin 2t & -\sin 2t \\ -2\sin 2t & \sin 2t \end{pmatrix};$$

$$DQ=\frac{1}{6}\begin{pmatrix} \cos 2t-e^t & \cos 2t+e^t \\ -\cos 2t-2e^t & -\cos 2t+2e^t \end{pmatrix}$$

$$DQC_0=\frac{1}{6}\begin{pmatrix} 4\cos 2t & -2\cos 2t \\ -4\cos 2t & 2\cos 2t \end{pmatrix}, \qquad QC_1=\frac{1}{6}\begin{pmatrix} 2e^t & 2e^t \\ 4e^t & 4e^t \end{pmatrix}$$

so that
$$DQC_0+QC_1=\frac{1}{3}\begin{pmatrix} 2\cos 2t+e^t & -\cos 2t+e^t \\ -2\cos 2t+2e^t & \cos 2t+2e^t \end{pmatrix}$$

Hence
$$x(t)=\frac{1}{3}\{(2\cos 2t+e^t)x(+0)+(e^t-\cos 2t)y(+0)\}$$
$$+\frac{1}{6}\{2\sin 2t\, Dx(+0)-\sin 2t\, Dy(+0)\};$$

$$y(t)=\frac{1}{3}\{(2e^t-2\cos 2t)x(+0)+(2e^t+\cos 2t)y(+0)\}$$
$$+\frac{1}{6}\{-2\sin 2t\, D\, x(+0)+\sin 2t\, D\, y(+0)\}.$$

5. *The determination of functions from their Laplace Transforms.* It is necessary for the solution of problems by the method of Operational Calculus to be able to determine the function $f(t)$ whose Laplace Transform $\phi(z)=L\{f(t)\}$ is in our possession. The following theorem is useful in this connection:

Let $\phi(z)$ be an analytic function and let α be a complex number (whose real part is positive) which is such that $z^\alpha\phi(z)$ is analytic at $z=\infty$ so that, near $z=\infty$,

$$\phi(z)=\frac{1}{z^\alpha}\sum_0^n \frac{c_n}{z^n}.$$

Then the power series $\displaystyle\sum_0^\infty \frac{c_n t^n}{\Gamma(n+\alpha)}$ *converges for every t and the function*

$$f(t)=t^{\alpha-1}\sum_0^\infty \frac{c_n t^n}{\Gamma(n+\alpha)}; \quad t>0$$

has, at points z whose real parts are sufficiently large, $\phi(z)$ for its Laplace Transform ($\Gamma(p)$ being the Gamma function which is defined by the formula

$$\Gamma(p)=\int_0^\infty e^{-t}t^{p-1}dt,\ R(p)>0.\quad \text{Thus } \Gamma(p+1)=p\ \Gamma(p)).$$

A proof of this theorem may be found in the book referred to in footnote[1].

On setting $\alpha=\frac{1}{2}$ we obtain the following special case of the theorem just given:

The function of which $\dfrac{1}{z^{\frac{1}{2}}}\left\{c_0+\dfrac{c_1}{z}+\dfrac{c_2}{z^2}+\cdots\right\}$ is the Laplace Transform is

$$f(t)=\frac{1}{\Gamma(\frac{1}{2})t^{\frac{1}{2}}}\left\{c_0+\frac{c_1t}{1/2}+\frac{c_2t^2}{1/2\cdot 3/2}+\cdots\right\}$$

In particular $L\left\{\dfrac{\cosh t^{\frac{1}{2}}}{t^{\frac{1}{2}}}\right\}=\dfrac{\Gamma(\frac{1}{2})}{z^{\frac{1}{2}}}e^{1/4z}=\left(\dfrac{\pi}{z}\right)^{\frac{1}{2}}e^{1/4z}$.

Again, on developing $(1+z^2)^{-\frac{1}{2}}$ near $z=\infty$ we obtain

$$L\{J_0(t)\}=\frac{1}{(1+z^2)^{\frac{1}{2}}}$$

where $J_0(t)$ is the familiar Bessel function of the first kind of zero order. Similarly on developing

$$\frac{1}{z(1+z^2)^{\frac{1}{2}}}$$

near $z=\infty$ we find that

$$L(\operatorname{erf} t^{\frac{1}{2}})=\frac{1}{z(1+z^2)^{\frac{1}{2}}}$$

where $\operatorname{erf} x$ is the *error function* defined by

$$\operatorname{erf} x=\frac{2}{\pi^{\frac{1}{2}}}\int_0^x e^{-t^2}\,dt.$$

On developing $\dfrac{e^{-1/z}}{z}$ near $z=\infty$ we find that

$$L\{J_0(2t^{\frac{1}{2}})\}=\frac{e^{-1/z}}{z}$$

and similarly we obtain the more general result

$$L\{t^{(\alpha-1)/2}J_{\alpha-1}(2t)^{\frac{1}{2}}\}=\frac{e^{-1/z}}{z^{\alpha}}, \quad R(\alpha)>0.$$

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We give here a small list of books selected from the many works which discuss operational caculus.

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COLLEGIATE ARTICLES

Original Papers Whose Reading Does Not Presuppose Graduate Training

A Novel Algorithm at the Freshman Level

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Introduction. It is a tedious task to tabulate the values of a polynomial such as

$$(1) \qquad 2x^4-4x^3-10x^2+11x+7$$

by using just the Remainder Theorem and synthetic division. The following algorithm avoids the copying and recopying of the coefficients and the determining of unused quotient coefficients.

Having determned -2 as a lower limit to the zeros of the polynomial (1), the work indicated below is performed:

(2)

2	-4	-10	11	7	
	-4	16	-12	2	
2	-8	6	-1	9	-2
	-2	10	-16		
2	-10	16	-17		-1
	0	0			
2	-10	16			0
	2				
2	-8				1

The reader will observe that the polynomial was divided by $x-(-2)$, the quotient by $x-(-1)$, etc., until the operation reached a natural conclusion. The work is continued as shown below:

(3)

				9	-2
				-17	
			-17	-8	-1
			32	15	
		16	15	7	0
		-24	-16	-1	
	-8	-8	-1	6	1
	8	0	-16	-17	
2	0	-8	-17	-11	2
	8	24	32	15	
2	8	16	15	4	3

In this process the rows opposite $-2,\ -1,\ 0,\ 1,\cdots$ are computed from the top down by multiplying the entry below the coefficient of x^k by k and adding the result to the entry in the row above and the next column to the right. The numbers $9,\ -8,\ 7,\ 6,\cdots$, which occupy the positions of remainders in synthetic division, are the values of the polynomial at $-2,\ -1,\ 0,\ 1,\cdots$. There is an instantaneous check at $x=0$, and a check by the usual method at $x=3$ checks that value and also indicates that the intermediate values are correct.

The writer has found this algorithm useful as bait to be thrown to promising undergraduates. It is apt to lead them to ideas not encountered in the usual courses and yet well within their reach. The following paragraphs give one explanation of the algorithm and indicate its connection with familiar problems.

1. *The construction of the algorithm.* In formal work in the elementary calculus of finite differences, the polynomial $x(x-1)\cdots(x-n+1)$

plays an important role and is designated by $[x]^n$. The usefulness of the symbol $[x]^n$ stems from the relation

$$(1.1) \qquad \Delta[x]^n = n[x]^{n-1},$$

where $\Delta f(x) = f(x+1) - f(x)$. To change a given polynomial

$$(1.2) \qquad f(x) \equiv a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$$

to the form

$$(1.3) \qquad b_0[x-\alpha]^n + b_1[x-\alpha]^{n-1} + \cdots + b_{n-1}[x-\alpha] + b_n,$$

note that b_n is the remainder when $f(x)$ is divided by $x-\alpha$ and the quotient $q_{n-1}(x)$ is $b_0[x-\alpha+1]^{n-1} + b_1[x-\alpha+1]^{n-2} + \cdots + b_{n-1}$. Thus b_{n-1} is the remainder when $q_{n-1}(x)$ is divided by $x-\alpha+1$ and all the numbers $b_{n-2}, b_{n-3}, \cdots, b_0$ may be so determined.*

(1.4) *The polynomial* $a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ *may be put in the form* $b_0[x-\alpha]^n + b_1[x-\alpha]^{n-1} + \cdots + b_{n-1}[x] + b_n$ by the *algorithm*:

$$\begin{array}{ccccc|c}
a_0 & a_1 & \cdots & a_{n-1} & a_n & \\
 & \alpha q_0 & \cdots & \alpha q_{n-2} & \alpha q_{n-1} & \\
\hline
q_0 & q_1 & \cdots & q_{n-1} & b_n & \alpha \\
 & (\alpha+1)q_0' & \cdots & (\alpha+1)q'_{n-2} & & \\
\hline
q_0' & q_1' & \cdots & b_{n-1} & & \\
 & & \cdots & & &
\end{array}$$

The numerical work (2) in the introduction has the effect of establishing the identity

$$2x^4 - 11x^3 - 10x^2 + 11x + 7$$
$$\equiv 2[x+2]^4 - 8[x+2]^3 + 16[x+2]^2 - 17[x+2] + 9.$$

The completing of the algorithm as carried out in (3) of the introduction is now easily explained.

For $f(x)$ in the form (1.3), $\Delta f(x)$ is given by

$$\Delta f(x) = nb_0[x-\alpha]^{n-1} + (n-1)b_1[x-\alpha]^{n-2} + \cdots + b_{n-1},$$

and $f(x+1) = f(x) + \Delta f(x) = b_0[x-\alpha]^n + (b_1 + nb_0)[x-\alpha]^{n-1} + \cdots$
$$+ (b_{n-1} + \alpha b_{n-2})[x-\alpha] + (b_n + b_{n-1}).$$

Hence, setting $x = x-1$,

$$(1.5) \qquad f(x) = b_0[x-\alpha+1]^n + (b_1 + nb_0)[x-\alpha+1]^{n-1} + \cdots$$
$$+ (b_{n-1} + 2b_{n-2})[x-\alpha+1] + (b_n + b_{n-1}).$$

*Whittaker and Robinson "The Calculus of Observations", Chap. I.

This means that if $f(x)$ is in the form (1.3) for a given α, it may be put in the same form for $\bar{\alpha}=\alpha+1$ by the equations:

$$\bar{b}_0=b_0;\quad \bar{b}_1=b_1+nb_0;\quad \cdots\quad ;\ \bar{b}_{n-1}=b_{n-1}+2b_{n-2};\quad \bar{b}_n=b_n+b_{n-1}.$$

Inspection of (3) of the introduction now shows that $2x^4-4x^3-10x^2+11x+7$ was put successively in the form (1.3) for $\alpha=-2, -1,0,1,\cdots$. The coefficients of these forms must be read *diagonally* from lower left to upper right and the last coefficient, b_n, must in each case be $f(\alpha)$.

2. *An inverse of the algorithm* 1.4. We will say a polynomial is in the a-form if it is in the form (1.2) and in the b-form if it is in the form (1.3) with $\alpha=0$. The algorithm (1.4) with $\alpha=0$ provides a method for going from the a-form to the b-form. What about an algorithm for going from the b-form to the a-form?

To set this up in a simple manner seems to require the following lemma:

(2.1) *If the numbers $B_0, B_1,\cdots, B_{k-1}, B_k$ are given in the identity*

$$B_0[x-1]^k+B_1[x-1]^{k-1}+\cdots+B_{k-1}[x-1]+B_k$$
$$\equiv b_0[x]^k+b_1[x]^{k-1}+\cdots+b_{k-1}[x]+b_k$$

the numbers $b_0,b_1,\cdots,b_{k-1},b_k$ may be computed by

$$\begin{array}{ccccc} B_0 & B_1 & \cdots & B_{k-1} & B_k \\ & -kb_0 & \cdots & -2b_{k-2} & -b_{k-1} \\ \hline b_0 & b_1 & \cdots & b_{k-1} & b_k \end{array}$$

Setting $\alpha=0$ and $n=k$ in the identity (1.5) yields

$$b_0[x]^k+b_1[x]^{k-1}+\cdots+b_{k-1}[x]+b_k$$
$$\equiv b_0[x-1]^k+(b_1+kb_0)[x-1]^{k-1}+\cdots+(b_{k-1}+2b_{k-2})[x-1]+(b_k+b_{k-1}).$$

Thus the identity (2.1) is equivalent to the equations

$$B_0=b_0;\quad B_1=b_1+kb_0;\cdots;\quad B_{k-1}=b_{k-1}+2b_{k-2};\quad B_k=b_k+b_{k-1},$$

or

$$b_0=B_0;\quad b_1=B_1-kb_0;\cdots;\quad b_{k-1}=B_{k-1}-2b_{k-2};\quad b_k=B_k-b_{k-1}.$$

The algorithm is merely a schematic solution of this system.

Now consider the problem of determining the numbers $a_n, a_{n-1},\cdots, a_1, a_0$ from the identity

(2.2) $$b_0[x]^n+b_1[x]^{n-1}+\cdots+b_{n-1}[x]+b_n$$
$$\equiv a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n.$$

Clearly $a_n=b_n$. Cancelling these and dividing both sides by x yields $b_0[x-1]^{n-1}+b_1[x-1]^{n-2}+\cdots+b_{n-1}\equiv a_0x^{n-1}+a_1x^{n-2}+\cdots+a_{n-1}$. Now by the algorithm of (2.1) this may be rewritten

$$b_0'[x]^{n-1}+b_1'[x]^{n-2}+\cdots+b'_{n-1}\equiv a_0x^{n-1}+a_1x^{n-2}+\cdots+a_{n-1}.$$

Thus $a_{n-1}=b'_{n-1}$ and there results a new identity

$$b_0'[x-1]^{n-2}+b_1'[x-1]^{n-3}+\cdots+b'_{n-2}\equiv a_0x^{n-2}+a_1x^{n-3}+\cdots+a_{n-2},$$

which may be transformed by (2.1) into

$$b_0''[x]^{n-2}+b_1''[x]^{n-3}+\cdots+b''_{n-2}\equiv a_0x^{n-2}+a_1x^{n-3}+\cdots+a_{n-2}.$$

Thus $a_{n-2}=b''_{n-2}$ and this process may be continued.

(2.3) *The algorithm for going from the b-form to the a-form is* :

$$\begin{array}{ccccc|l}
b_0 & b_1 & \cdots & b_{n-2} & b_{n-1} & b_n=a_n \\
 & -(n-1)b_0^1 & & -2b'_{n-3} & -b'_{n-2} & \\
\hline
b_0' & b_1' & \cdots & b'_{n-2} & \multicolumn{1}{|l}{b'_{n-1}=a_{n-1}} & \\
 & -(n-2)b_0'' & & -b''_{n-3} & \multicolumn{1}{|l}{} & \\
\hline
b_0' & b_1'' & \cdots & \multicolumn{1}{|c|}{b''_{n-2}=a_{n-2}} & & \\
 & & \cdots\cdots\cdots & & &
\end{array}$$

For example, $[x]^5+15[x]^4+65[x]^3+90[x]^2+31[x]+1$ is coverted into $x^5+5x^4+10x^3+10x^2+5x+1$ by

1	15	65	90	31	1
1	11	32	26	5	
1	8	16	10		
1	6	10			
1	5 .				

Here, and in later numerical examples, the intermediate rows are deleted.

3. *Two applications. The sum of a finite series whose general term is a polynomial.*

The formal relation $\Delta[x]^n=n[x]^{n-1}$ means that a polynomial in the b-form may be summed easily. Consider the sum defined for any positive integer x:

$$S(x)=\sum_{i=1}^{x} f(i). \tag{3.1}$$

Clearly $\Delta S(x)=f(x+1)$ and $f(x+1)$ may be put in the b-form by

the algorithm (1.4). From $f(x+1)=b_0[x]^n+b_1[x]^{n-1}+\cdots+b_{n-1}[x]+b_n$ we conclude that

$$S(x)=b_0[x]^{n+1}/(n+1)+b_1[x]^n/n+\cdots+b_{n-1}[x]^2/2+b_n[x]+c.$$

Since $S(1)=f(1)=b_n$, we conclude that $c=0$ and

$$S(x)=\frac{b_0[x]^{n+1}}{n+1}+\frac{b_1[x]^n}{n}+\cdots+\frac{b_{n-1}[x]^2}{2}+b_n[x]. \tag{3.2}$$

It would be easy in a numerical case to use the algorithm (2.3) to put the result in the usual a-form.

Undergraduates are frequently fascinated by formulas for the sums of the power of the integers from 1 to n. A particular case of such a problem is worked out below by the method outlined above.

$$S_5(n)=\sum_{i=1}^{n} i^5$$

$$\begin{aligned}\Delta S_5(n)&=n^5+5n^4+10n^3+10n^2+5n+1\\&=[n]^5+15[n]^4+65[n]^3+90[n]^2+31[n]+1.\end{aligned}$$

$$\begin{aligned}S_5(n)&=\frac{[n]^6}{6}+3[n]^5+\frac{65[n]^4}{4}+30[n]^3+\frac{31[n]^2}{2}+[n]\\&=\frac{1}{12}(2[n]^6+36[n]^5+195[n]^4+360[n]^3+186[n]^2+12[n])\\&=\frac{1}{12}(2n^6+6n^5+5n^4-n^2).\end{aligned}$$

$$\begin{array}{ccccc|c}1&5&10&10&5&1\\\hline 1&6&16&26&\multicolumn{1}{|c}{31}&\\\hline 1&8&32&\multicolumn{1}{|c}{90}&&\\\hline 1&11&\multicolumn{1}{|c}{65}&&&\\\hline 1&15&&&&\end{array}$$

$$\begin{array}{cccccc|c}2&36&195&360&186&12&0\\\hline 2&26&91&87&12&\multicolumn{1}{|c}{0}&\\\hline 2&18&37&13&\multicolumn{1}{|c}{-1}&&\\\hline 2&12&13&\multicolumn{1}{|c}{0}&&&\\\hline 2&8&\multicolumn{1}{|c}{5}&&&&\\\hline 2&6&&&&&\end{array}$$

Interpolation. The algorithm of (2) and (3) in the introduction enables one to compute the values of a polynomial $f(x)$ at a set of evenly spaced values of x. If on the other hand, the values of a polynomial of degree n are given at $n+1$ evenly spaced values, the coefficients of $f(x)$ may be computed by reversing the operation.

The algorithm for changing from the b-form to the a-form fits in nicely with the usual theory. If $f(x)$ is represented by the Gregory-Newton formula

$$(3.3) \qquad f(x)=\frac{\Delta^n f(0)}{n!}[x]^n+\frac{\Delta^{n-1}f(0)}{(n-1)!}[x]^{n-1}+\cdots+\Delta f(0)[x]+f(0),$$

where $\Delta^n f(0)=\left[\Delta^n f(x)\right]_{x=0}$, then the b-form of the polynomial is known, once the successive differences at zero are computed from a table of values. The a-form then follows easily by the algorithm (2.3). An example follows:

Given the table of values $f(x)=3,10,17,33,91$ at $x=0,1,2,3,4$ the differences may be computed as below:

$$\begin{array}{ccccccccc}
3 & & 10 & & 17 & & 33 & & 91 \\
& 7 & & 7 & & 16 & & 58 & \\
& & 0 & & 9 & & 42 & & \\
& & & 9 & & 33 & & & \\
& & & & 24 & & & &
\end{array}$$

$$\begin{array}{cccc|c}
2 & 3 & 0 & 14 & 6 \\
\hline
2 & -3 & 6 & \multicolumn{1}{|c}{8} & \\
\cline{1-3}
2 & -7 & \multicolumn{1}{|c}{13} & & \\
\cline{1-2}
2 & -7 & & &
\end{array}$$

where $f(0)$, $\Delta f(0),\cdots$ are 3,7,0,9,24

Thus $$f(x)=\frac{24}{4!}[x]^4+\frac{9[x]^3}{3!}+\frac{7[x]}{1!}+3$$

$$=\tfrac{1}{2}\left(2[x]^4+3[x]^3+14[x]+6\right)$$

$$=\tfrac{1}{2}(2x^4-9x^3+13x^2+8x+6).$$

Southern Methodist University.

CURRENT PAPERS AND BOOKS

Edited by
H. V. CRAIG

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

The purpose and policies of the first division of this department (Comments on Papers) derive directly from the major objective of the MATHEMATICS MAGAZINE which is to encourage research and the production of superior expository articles by providing the means for prompt publication.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited. Comments which express conclusions at variance with those of the paper under review should be submitted in duplicate. One copy will be sent to the author of the original article for rebuttal.

Communications intended for this department should be addressed to

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Sequential Analysis of Statistical Data: Applications. Prepared by the Statistical Research Group, Columbia University for the Applied Mathematics Panel, National Defense Research Committee, Office of Scientific Research and Development. Columbia University Press, New York, September, 1945. $6.25.

This collection of seven stapled pamphlets in a ring binder deals with a new statistical technique, that of sequential analysis, introduced in 1943 by A. Wald for use in war research. An earlier "Restricted" edition for limited circulation preceded this printing and has, no doubt, served to reduce errors and misprints common to photographed typewritten copy.

No attempt is made to present details of the mathematical theory. Indeed, in the appendix which is mathematical in nature the following statement occurs: "This appendix covers only a small part of the theory covered in SRG 75, and that part is covered briefly and intuitively. Any mathematician who may stray into this Appendix should be assured that the validity of the conclusions in no case depends upon the type of reasoning presented here." By way of explanation, SRG 75 is Statistical Research Group report number 75, *Sequential Analysis of Statistical Data: Theory* by A. Wald, September, 1943. A more accessible paper for the general reader is Wald's paper, "Sequestial Tests of Statistical Hypotheses," *Annals of Mathematical Statistics* Vol. 16 (1945) part 2. The mathematician is therefore urged to read this paper and similar research articles for proofs.

The style is such that users who are neither experts in mathematics nor statistics can apply the methods developed to any one of the specific applications described. Thus, the second pamphlet deals with a binomial distribution. Suppose, for example, that a group or lot of a particular type product is being examined to see if it is a good or a bad lot. In many cases, the test might be destructive to an individual item, or even if non-destructive too costly to apply to each individual item. Prior to the invention of sequential analysis it was customary to select a sample of size N, test these N items, and then either accept or reject the lot from which the sample was drawn.

In sequential analysis, the following procedure is used: After each observation, one of the following three steps is taken:

(1) the lot is accepted, or
(2) the lot is rejected, or

(3) judgment is deferred and another observation is made. The nature of the theory is such that good lots are quickly accepted, i. e. alternative (1) can be accepted before N trials which would have been necessary under the old procedure, bad lots are quickly rejected, but border-line lots still require extensive testing, perhaps more than N items.

If the probability of rejecting a lot with p_1 proportion of bad items is α and if the probability of accepting a lot with p_2 proportion of bad items is β, then Wald found that the criteria for choices (1) and (2) above are

$$(1) \qquad \frac{p_1{}^d(1-p_1)^{n-d}}{p_2{}^d(1-p_2)^{n-d}} \geq \frac{1-\alpha}{\beta} \quad \text{for acceptance,}$$

$$(2) \qquad \leq \frac{\alpha}{1-\beta} \quad \text{for rejectance,}$$

where n is the number of items tested to date and d is the number of bad items. Testing continues until one of these relations is satisfied.

Both numerical and graphical procedures useful in keeping track of the progress of the test are explained, and tabular data and nomographic charts with full explanations for their use are given.

The removal of the "Restricted" tag from this statistical problem is a welcome move and although the volume under review is essentially non-mathematical, the related mathematical portions will enrich the literature of mathematical statistics.

The University of Texas. ROBERT E. GREENWOOD.

The Magic of Numbers. E. T. Bell. New York: Whittlesey House, McGraw-Hill, 1946. X+418. (Frontispiece). $3.50.

While considering Pythagoras' numerology Bell repeatedly asks the question were numbers discovered or invented? The book doesn't reach a definitive answer to the question but the case for discovery seems more plausible after the reader has plowed through the story and met on the way such variant personalities asThales, the fabulous Empedocles, Pythagoras and his Brotherhood, Plato, Dante, Bacon, Galileo, Newton, Berkeley, Saccheri, Lobachewsky and Kant. Integrated all this forms a background for the controversy generated by Sir Arthur Eddington from his claim that the fundamental physical laws can be deduced in an epistemological manner without recourse to observation or experiment. This controversy receives only limited space in the last chapter but deserves much more. Being in the foundations of physical science it is of transcendent importance in this atomic age.

A highlight of the book is the passage where Bell considers an infant subjectively learning Boolean algebra in the relation I and not $-I$. The book is written so only ordinary arithmetic is prerequisite to its reading, the mathematics of the book being limited to the beginnings of the theory of numbers while touching upon formal logic and non-euclidean geometry.

For reference while reading the book it is well to have Runes' *Dictionary of Philosophy* on hand. Helpful background information on almost everything from Absolute to Zeno is thus available and Alonzo Church's formal logic definitions are outstanding.

For more information on Eddington's ideas on epistemology one should consult *The Philosophy of Physical Science*, 1939.

Colorado Springs, Colorado. DUANE STUDLEY.

HISTORY AND HUMANISM

Edited by
G. WALDO DUNNINGTON and A. W. RICHESON

Papers on the history of Mathematics *per se*, the part it has played in the development of our present civilization and its relation to other sciences and professions are desirous for this department.

A French Mathematician of the Sixteenth Century Jacques Peletier (1517-1582).

by V. THEBAULT
Tennie (Sarthe) France

Biography. Jacques Peletier, French writer, poet and mathematician, was born at Le Mans, July 25, 1517. He was sent at an early age to Paris and there he attended Navarre college, where his brother Jean taught philosophy. Soon after leaving college he prepared to become a teacher. In 1544 he obtained through the influence of the bishop of Le Mans,René du Bellay, the position of tutor of mathematics at the college of Bayeux. Three years later, in 1547, he was at the head of this school, when he delivered from the puplit of Notre Dame the funeral oration of Henry VIII of England. This event seems to have been a turning point in his life.

The restlessness of Peletier prompted him to resign, in order to go to live in the house of the printer, Vascosan, where he conceived the curious project of reforming spelling according to pronunciation.* This forms the subject matter of his *Dialogue on French Spelling and Pronunciation*, in two volumes.

Peletier had scarcely completed this work when he became deeply interested in medicine, which he went to study in Poitiers in 1550. He stayed successively in Bordeaux, Lyon and Rome. He then seemed to be tired of wandering, and settled down a while in Paris, where he received his Licentiate in medicine. The turmoil of the Wars of Religion again took him away from Paris. After a long stay at Annecy, Savoy, he returned to Paris in 1578 to become the head of the college of Le Mans.

Since he had always remained faithful to the Church of Rome, in an era when religious differences menanced French national unity, the Faith that he had inherited from family tradition caused him, toward the end of his life, to humbly implore divine aid in contemplating the

**Dialogue de l Orthografe e Prononciation francoese.* Marnef, Poitiers, 1550.

eternal laws which govern numbers and figures. It is in his beloved geometric language that he phrased his last prayer:

> Thou, One, All Infinite, Thou the Circle and the Center,
> Whence comes every line, where every line returns,
> Make me end in Thee, I, who in Thee began.

Peletier died in Paris in July, 1582.

The Mathematical Work of Peletier. To understand the meaning of the first mathematical work published by Peletier we must take into account the extreme poverty of Western thought in the domain of the exact sciences up to the sixteenth century. After the fall of the Western Empire all that remained of the science of Euclid, Archimedes and appollonius consisted of meager summaries, simple enumerations of the most elementary rules transmitted by Boèce and Cassiodore. Arithmetic and geometry were reduced to statements covering a few applications. Writers on the subject of algebra, which had come from India by way of the Arabs, were reluctant to solve problems in a general manner; they repeated the reasoning for each particular problem. Furthermore, no symbolism had been adopted which permitted a common treatment of all numbers, positive or negative, integers or fractions, real or complex.

From 1515 on, the wars in Italy had brought French scientists into contact with the students of the learned Greeks who, chased from Byzantium sixty years before by the Turks, had fled to Sicily and had gone little by little toward the North, bringing with them the texts of the great mathematicians of the Alexandrian period. It is difficult to depict the astonishment of Europeans upon reading such majestic and clear monuments of pure logic as are, for instance, the *Elements* of Euclid, the *Conics* of Appollonius and the *Treatise on Spirals* of Archimedes. Thus Peletier, who doubtless had never known of the existence of such treatises before his arrival in Paris, gave himself no respite until he had learned Greek by himself (Greek was not taught in the schools of France until 1532) and had read the original texts.

For the study and teaching of arithmetic he had at his disposal only the manual published several years before by Gemme of Friesland, professor at the University of Louvain. He corrected many of the errors which abounded in the original edition, and, encouraged by his students, published in 1545 a first Latin edition. Its success was considerable, for, a rare event for this period, Peletier's *Arithmetic* was reprinted, without change, but in French, at Poitiers in 1548 and 1552, at Lyon in 1554 and 1570, in Latin in 1563, and finally in Italian in 1567.

During this time Peletier undertook the study of algebra. Probably he was the first in France to publish a treatise in algebra, in 1554 inaugurating the use of literal symbols, which were later made popularly

Viète. He showed that quadratic equations admit not one, but two roots, whose product equals the constant term. This was extended later by Viète to cubic equations and by Harriot to equations of arbitrary degree. Also he carried over to biquadratic equations what he had proved for quadratic equations.

By publishing his *Algebra* Peletier became a leader and a popularizer, but in spite of the clarity of his exposition he was unable to overcome the hostility brought upon him by his use of the French language. But he was of peaceful temperament, and the only revenge he took was to translate the second edition of his *Algebra* (1560) into Latin, with the ironical title, *De occult a parte numerorum quam Algebram vocant.*

After 1554 Peletier, without neglecting the new editions of his works, devoted himself more and more to consolidating the theory. At the same time his enthusiasm for Greek mathematics began to wane After Cardan had called his attention to flaws in the majestic edifice of Euclid's *Elements*, he examined closely the foundations of geometry, and believed, quite some time before Lobachewsky and Riemann, that he had discovered defects.

Peletier reduced to two the paralogisms and contradictions which he believed he had found in Euclid's *Elements*, at that time the holy book of mathematicians. The first referred to the proof of the congruence of figures by superposition and the second to the magnitude of the angle formed by a curved line and its tangent. For the first question he let himself be convinced by Jean Buthéon or Borel. But for the second question he maintained his position.

In the thirteenth proposition of the third book of his *Elements*, Euclid considers the figure formed by a circle and one of its tangent lines. By analogy with the angle formed by two intersecting straight lines, he considers the angle of contingence and shows that it is smaller than any assignable angle.

Peletier states: "When I examined this proposition more carefully, it suddenly appeared to me that geometry could not be satisfied with it without admitting inconsistency. First, we can not suppose that there can exist a quantity smaller than all others, as this angle of contingence or contact would be, for quantity is what is formed of parts, and it is only with respect to quantity that things are said to be equal or unequal. On the other hand, continuous quantities are indefinitely divisible. This is why it seems to me impossible to reconcile what is said here about the angle of contingence with the first proposition of the tenth book."

However, the disturbance stirred up by these discussions as to the angle of contingence kept increasing. Clavius of Bamberg, a noted

astronomer, had just published in Rome a treatise on geometry in which he described the ideas of Peletier as sophisms and hallucinations. Peletier's answer only served to spread controversy, as the whole Jesuit order backed Clavius, himself a Jesuit. It was to last long after Peletier's death and maybe it would still be going on if Newton himself had not definitely settled the argument in favor of Peletier by calling attention to the possibility of variation of curvature near the point of contact.

This controversy caused Peletier, just a few months before his death, to write a short 14-page memoir entitled *De contactu Linearum*, of which only one copy remains. But this work, by the precision of its definitions, the clarity of its reasoning, its careful and well ordered exposition, belonged already to the Cartesian era, fifty years ahead of contemporaneous mathematicians.

Peletier, who had understood perfectly the core of the difficulty, analyzed at length the notion of angle. He called attention first to the fact that since geometry is the science of magnitudes it can have no other object than the study of quantity. Since angles properly belong to it, angles must be measurable quantities. Their measure can be sought neither in the length of the sides, which can be extended without changing the angle nor in the area included between the sides, since this area is either unbounded or is that of a triangle if a suitable third line is drawn. The magnitude of an angle must exist at the vertex itself and it will be smaller the more nearly the sides coincide.

Going from this to the notion of contact Peletier notes that the angles formed by homologous sides of two similar figures are equal. It follows that angles of contingence, at least those that circles of different radii form with a common tangent line, can not have different magnitudes. Therefore angles of contact are not quantities and cannot, whatever Euclid may have said, be compared to angles formed by intersecting lines.

In conclusion let us cite the following lines written by Abbé M. Thureau in his book entitled *Jacques Peletier, Sixteenth Century Mathematician of Le Mans*, 1934: "In all the abundant scientific literature of the Rennaissance I do not know of a single work which even nearly approximates Peletier's *De contactu linearum*, a masterpiece incontestably worthy of appearing in an anthology of mathematics if, as I hope, one is ever published."

NOTE: In the time of provincial France the city of Le Mans, the capital of the province of Maine situated in the west of France, was the birthplace of another illustrious mathematician, the Reverend Marin Mersenne, born September 8, 1588 in a town only a little distance from Le Mans. He died in Paris, in the convent of the Minimes, on rue Royale, September 1, 1648, in the presence of his friend Gassendi. He was permanent secretary of the Academy of Sciences at its beginning. His life and works are well known.

Translated by Miss Marie Edith Byrne.

TEACHING OF MATHEMATICS

Edited by
JOSEPH SEIDLIN, L. J. ADAMS and C. N. SHUSTER

This department is devoted to the teaching of mathematics. Thus articles on methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, *as a teacher*, are interested, or questions which you would like others to discuss, should be sent to JOSEPH SEIDLIN, Alfred University, Alfred, New York.

Proofs of the Fundamental Theorems of Spherical Trigonometry

by C. E. CLARK

This note presents derivations of some fundamental formulas for the spherical triangle. The derivations are shorter and simpler than those found in the textbooks. Furthermore, the first proof enables a teacher to present the essential applications of spherical trigonometry after less than one lecture on the spherical triangle. The formulas derived are the law of cosines for sides, the law of sines, the law of cosines for angles, and Napier's rules. The derivations are shorter and simpler than those given in the textbooks for the following reasons. The use of solid geometry including the theory of the polar triangle is avoided. The only formulas from plane trigonometry used are the law of cosines, the reciprocal relations, and the Pythagorean relations. In no case is a proof divided into two or more cases depending upon the quadrants of the angles involved. Another important feature of the discussion is that it presents a brief course in spherical trigonometry. This brief course arises from the fact that the first derivation is for the law of cosines for sides. This law alone will handle applications of spherical trigonometry to the terrestrial and celestrial spheres.

To derive the law of cosines for sides we denote the sides of a spherical triangle as well as their measures by a, b, and c. The opposite vertices and their measures are A, B, and C, respectively. The center of the sphere is O. A plane is constructed perpendicular to OA. This plane meets OA, OB, and OC in the distinct points, P, Q, and R, respectively. Application of the law of cosines for plane triangles to the triangles PQR and OQR gives the equation

$$(PQ)^2+(PR)^2-2(PQ)(PR)\cos A=(OQ)^2+(OR)^2-2(OQ)(OR)\cos a.$$

If no side or angle of the spherical triangle has the measure 90°, the lengths PQ, PR, OQ, and OR are replaced by $OP \tan c$, $OP \tan b$, $OP \sec c$, and $OP \sec b$, respectively. A simple reduction gives the law of cosines in the form

$$(1) \qquad \cos a = \cos b \cos c + \sin b \sin c \cos A.$$

That this relation still holds when a measurement is 90° is clear on the grounds of continuity. This law of cosines gives three independent equations which will solve any spherical triangle. In some cases the solution would be long, and the elimination of extraneous solutions would be tedious. However, when three sides or two sides and the included angle are given, another measurement can be found readily by the law of cosines. These cases cover several of the significant applications. Examples of such applications are the problem of finding distance and bearing when the latitude and longitude of two points on the earth's surface are given, and the problem of finding the time when the latitude of an observer, altitude of the sun, and declination of the sun are given.

We turn next to the law of sines. A geometric derivation resembling the above derivation of the law of cosines is given in the Encyclopaedia Britannica, 14th edition, volume 22, p. 474. This encyclopaedia article uses a few results from solid geometry and is not quite as simple as the above derivation of the law of cosines. However, it may be preferred to the analytic derivation which we now present. For reference, we write the following two statements of the law of cosines:

$$(2) \qquad \cos b = \cos c \cos a + \sin c \sin a \cos B,$$

$$(3) \qquad \cos c = \cos a \cos b + \sin a \sin b \cos C.$$

Multiplication of both sides of (1) and (2) by $\cos a$ and $\cos b$, respectively, and subtraction gives the equation

$$(4) \qquad \cos^2 a - \cos^2 b = (\sin b \cos a \cos A - \sin a \cos b \cos B)\sin c.$$

Similarly, multiplication of (1) and (2) by $\sin a \cos B$ and $\sin b \cos A$, respectively, leads to the equation

$$(5) \qquad \sin a \cos a \cos B - \sin b \cos b \cos A$$
$$= (\sin a \cos b \cos B - \cos a \sin b \cos A)\cos c.$$

Squaring both sides of (4) and (5), adding, and cancelling one term from each side of the resulting equation, we get the equation

$$(\cos^2 a - \cos^2 b)^2 + \sin^2 a \cos^2 a \cos^2 B + \sin^2 b \cos^2 b \cos^2 A$$
$$= \sin^2 b \cos^2 a \cos^2 A + \sin^2 a \cos^2 b \cos^2 B.$$

This equation can be written

$$(\cos^2 a-\cos^2 b)^2+\sin^2 a\cos^2 B(\cos^2 a-\cos^2 b)$$
$$-\sin^2 b\cos^2 A(\cos^2 a-\cos^2 b)=0.$$

Removing the common factor and changing cosines into sines we obtain the equation

$$\sin^2 a\sin^2 B=\sin^2 b\sin^2 A.$$

Since all angles involved are between 0° and 180°, we obtain the law of sines.

The law of cosines for angles can be obtained from the theory of the polar triangle. However, the geometry involved can be avoided by the following analytic proof. The law in question is that

$$\cos C=\cos A\cos B+\sin A\sin B\cos c. \tag{6}$$

To show that this equation is an identity, the law of sines is used to show that

$$\sin A\sin B=\frac{\sin a\sin b\sin^2 C}{\sin^2 c}$$

$$=\frac{\sin a\sin b}{\sin^2 c}(1-\cos^2 C).$$

This last expression is substituted in (6). In the resulting equation $\cos C$, $\cos A$, and $\cos B$ are replaced by the expressions obtained from (3), (1), and (2), respectively. Then obvious simplifications complete the proof that (6) is an identity.

We can now obtain Napier's rules from the above results. From (6) we see that

$$\cos A=-\cos B\cos C+\sin B\sin C\cos a, \tag{7}$$

$$\cos B=-\cos C\cos A+\sin C\sin A\cos b. \tag{8}$$

Henceforth we assume that $C=90°$. From (3) we see that $\cos c=\cos a\cos b$. From (7) and (8) we get the relations $\cos A=\sin B\cos a$ and $\cos B=\sin A\cos b$. From (6) we obtain the equation $\cos c=\cot A\cot B$. The law of sines gives the relations $\sin a=\sin c\sin A$ and $\sin b=\sin c\sin B$. Using the relations already derived we see that

$$\tan a=\frac{\sin a}{\cos a}$$

$$=\frac{\sin c \sin A \sin B}{\cos A}$$

$$=\sin b \tan A.$$

From symmetry we have that $\tan b=\sin a \tan B$. Furthermore,

$$\tan a=\frac{\sin a}{\cos a}$$

$$=\frac{\tan b \cos b}{\tan B \cos c}$$

$$=\frac{\sin b}{\tan B \cos c}$$

$$=\frac{\sin c \sin B}{\cos c \tan B}$$

$$=\tan c \cos B.$$

Symmetry gives the equation $\tan b=\tan c \cos A$. Thus Napier's ten rules are derived.

Napier's rules can be derived from the law of cosines for sides and the law of cosines for angles without use of the law of sines. Hence Napier's rules can be obtained by first proving the law of cosines for sides, next obtaining the law of cosines for angles by the theory of the polar triangle, and finally deriving the rules in the following way. Four of the rules are obtained as the first four above. Using these four results we have

$$\cos A=\frac{\cos a-\cos b \cos c}{\sin b \sin c}$$

$$=\frac{\cos a-\cos a \cos^2 b}{\sin b \sin c}$$

$$=\frac{\cos a \sin^2 b}{\sin b \sin c}$$

$$=\frac{\cos a \sin b}{\sin c}$$

$$=\frac{\cos c \sin b}{\cos b \sin c}$$

$$=\cot c \tan b.$$

This gives the rule $\tan b = \tan c \cos A$, and using symmetry $\tan a = \tan c \cos B$. Next we have

$$\begin{aligned}\sin a &= \tan a \cos a\\ &= \frac{\tan c \cos B \cos A}{\sin B}\\ &= \cot B \tan b.\end{aligned}$$

This is the rule $\tan b = \sin a \tan B$. From symmetry we have that $\tan a = \sin b \tan A$. The remaining two rules are obtained from the equalities

$$\begin{aligned}\sin a &= \tan b \cot B\\ &= \frac{\tan b \cos c}{\cot A}\\ &= \tan c \cos A \cos c \tan A\\ &= \sin A \sin c.\end{aligned}$$

Symmetry gives $\sin b = \sin B \sin c$.

Emory University.

PROBLEMS AND QUESTIONS

Edited by
C. G. JAEGER and H. J. HAMILTON

This department will submit to its readers, for solution, problems which seem to be new, and subject-matter questions of all sorts for readers to answer or discuss, questions that may arise in study, research or in extra-academic applications.

Contributions will be published with or without the proposer's signature, according to the author's instructions.

Although no solutions or answers will normally be published with the offerings, they should be sent to the editors when known.

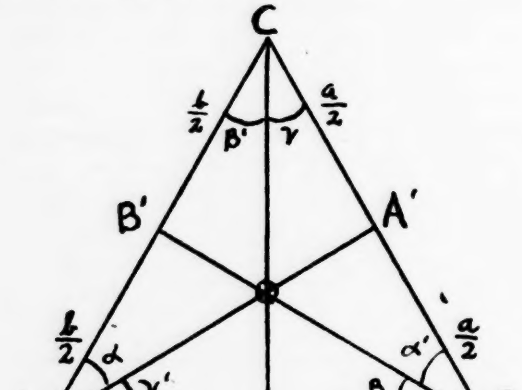
Send all proposals for this department to the Department of Mathematics, Pomona, College Claremont, California.

SOLUTIONS

Following is a solution to Problem No. 3 (Vol. XXI, No. 1, p. 58) proposed by *Nev. R. Mind.*

Solution by *Laurence A. Ringenberg,* Charleston, Ill.

To prove: If the medians of a triangle are proportional to the corresponding sides, the triangle is equilateral.



Let $AA'=a'$, $BB'=b'$, $CC'=c'$:

Given $a:b:c=a':b':c'$. It follows that

$$\frac{a}{2}:\frac{b}{2}:\frac{c}{2}=\frac{2a'}{3}:\frac{2b'}{3}:\frac{2c'}{3}=\frac{a'}{3}:\frac{b'}{3}:\frac{c'}{3}$$

Hence $\Delta AOB' \sim \Delta BOA'$ and $\alpha=\alpha'$. Similarly $\beta=\beta'$ and $\gamma=\gamma'$. It follows that $\alpha+\beta+\gamma=\pi/2$; $\alpha+\beta+\gamma'+AB'O=\pi$; $AB'O=\pi/2$. Similarly,

$BA'O$ and $BC'O$ are right angles. It follows that $\Delta AB'B \cong \Delta CB'B$ and that $<A=<C$. Similarly $<C=<B$. Thus the triangle is equiangular and hence also equilateral.

Solution for Problem 4, proposed by *Pedro A. Piza*, San Juan, Puerto Rico:

Solution by *Paul A. Clement*, University of California at Los Angeles.

Multiplying $a^2+b^2=c^2$ by a^2-b^2, we get $a^4-b^4=c^2a^2-c^2b^2$, which gives the desired relation $a^4+c^2b^2=b^4+c^2a^2$. Furthermore, $c^2b^2+a^4 =c^4-c^2a^2+a^4=c^4a^2b^2$, and it can be noted that these relations are valid for the sides of *any* right triangle.

Taking $c=a+1$, we find that each function considered above, when augmented by 3 is factorable to an *algebraic* square:

$$\begin{aligned} a^2c^2+b^4+3 &= a^2c^2+(c^2-a^2)^2+3 \\ &= a^2c^2+(c+a)^2+3 \\ &= a^2c^2+2ac+1+c^2+a^2+2 \\ &= (ac+1)^2+2a^2+2a+2+1 \\ &= (a^2+a1)^2+2(a^2+a+1)+1 \\ &= (a^2+a+2)^2 \end{aligned}$$

Since b and c are integers when a is, we have that $a^2c^2+b^4+3$ is a perfect square when $c=a+1$.

Also solved by *H. E. Bowie*, Indian Orchard, Mass.; *Laurence A. Ringenberg*, Charleston, Ill.; *Thomas Griselle*, Hollywood, Cal.

No. 600 (N. M. M.). Proposed by *N. A. Court*, University of Oklahoma.

Three concurrent non-coplanar lines, bearing the equal segments SA, SB, SC, are cut in the points A', B', C' by a variable plane $PA'B'C'$ passing through a fixed point P. Find the locus of the points common to the three variable spheres having A', B', C' for centers and AA', BB', CC', for radii

I Solution by *Howard Eves*, College of Puget Sound, Tacoma, Washington

Let u, v, w, r, s, t, p be position vectors of A, B, C, A', B', C', P respectively with respect to S as origin. Then we have

$$u \cdot u = v \cdot v = w \cdot w = k, \text{ say,}$$

where k is a scalar. Let x be the position vector of a point common to to the three spheres $A'(A'A)$, $B'(B'B)$, $C'(C'C)$. Then, by the conditions of the problem, we must have

$$(1) \qquad |p-r,\ p-s,\ p-t|=0,$$

$$(2) \qquad x\cdot x-2x\cdot r=k-2u\cdot r,$$

$$(3) \qquad x\cdot x-2x\cdot s=k-2v\cdot s,$$

$$(4) \qquad x\cdot x-2x\cdot t=k-2w\cdot t,$$

$$(5) \qquad r=au,\ s=bv,\ t=cw,$$

where a, b, c are scalars Substituting (5) in (2), (3), (4) and solving for a, b, c we find

$$(6) \qquad a=\frac{x\cdot x-k}{2(x\cdot u-k)},\quad b=\frac{x\cdot x-k}{2(x\cdot v-k)},\quad c=\frac{x\cdot x-k}{2(x\cdot w-k)},$$

Substituting (5) and (6) in (1) we find, after expanding and then removing the non-vanishing factor $(x\cdot x-k)^2$,

$$(7) \qquad (x\cdot x|u,v,w|-2x\cdot[u|p,v,w|+v|u,p,w|+w|u,v,p|] \\ +k[2|p,v,w|+2|u,p,w|+2|u,v,p|-|u,v,w|]=0$$

Thus the required locus is seen to be a sphere whose center and radius may readily be found from its equation (7).

The analagous problem in euclidean n-space may be similarly solved, yielding an n-sphere for the required locus

II Solution by *Nev. R. Mind.*

The tangent planes α,β,γ to the three variable spheres (A'), (B'), (C') at the fixed points A, B, C are also tangent, at those points, to the fixed spheres (S) having S for center and $SA=SB=SC$ for radius, i. e., the three planes are the radical planes of the pairs of spheres (S), $(A)'$; (S), (B'); (S), (C'), respectively. Thus the fixed point $L=\alpha\beta\gamma$ is the radical center of these four spheres, and their orthogonal sphere (L) is the fixed sphere having L for center and orthogonal to the fixed sphere (S).

The radical axis of the three spheres (A'), (B'), (C') passes through the point L and meets the three spheres in the same two points M, M', inverse with respect to the sphere (L). On the other hand, the points M, M' a e symmetrical with respect to the plane of centers $A'B'C'$ of the three spheres, these points are therefore equidistant from the fixed point P through which the plane $A'B'C'$ passes, by assumption. Thus M, M' lie on the sphere (P) having P for center and $PM=PM'$ for radius, and the sphere (P) is orthogonal to (L). Now the fixed point P

is the center of only one sphere orthogonal to the fixed sphere (L), hence (P) is the locus of the points M, M'.

BIBLIOGRAPHICAL NOTE. The corresponding question in the plane was considered in the *Nouvelles Annales de Mathématiques*, series 2, Vol. 17, 1878, p. 523, Q. 1286.—N. A. C.

PROPOSALS

13. Proposed by *Julius Sumner Miller*, Dillard University, New Orleans, La.

A uniform chain of length L and total mass M is held vertically with its lower end just touching a platform balance. The fixed upper end is released and the chain "accumulates" on the scale pan. What is the maximum reading of the balance?

14. Proposed by *A. K. Waltz*, Clarkson College, Potsdam, N. Y.

The tangent of an angle, whose sides slide on two given circles, has constant magnitude. Determine and discuss the locus of its vertex.

15. Proposed by *Victor Thébault*, Tennie, Sarthe, France.

A rectangular segement EF of constant length slides on the diameter $AB=2R$ of a semicircle (O). The perpendiculars to AB at E and F meet (O) at M and N. Show that the points of intersection P and P' of the circles passing respectively through M and N and having their centers at fixed points C and D on AB are on a circle concentric to (O), and that, if $\overline{CD}=R$, we have the relation

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$$

between the angles α,β,γ made by AM, BN, AQ with AB, Q being the point where the circle of center C and radius $\overline{CA}$ intersects the locus of P.

16. Proposed by *H. E. Bowie*, American International College.

A circle of radius 3-in. is tangent externally to a rectangle at the mid-point of one end. Two other circles, both of radius 2-in., are tangent externally to the two sides of the rectangle and to the first circle. The rectangle is 2-in. wide. Find the radius of a circle to which the three given circles are tangent internally

17. Proposed by *Pedro A. Piza*, San Juan, P. R.

Let x, y and z be any three arbitrary non-zero integers, and let p be

a prime number greater than 2. Prove that for any value of p whatsoever

$$[(x+y)(x+z)]^p-[x(x+y+z)]^p-[yz]^p=pxyz(x+y)(x+z)(x+y+z)K$$

where K is always an integer.

18. Proposed by *Anonymous.*

Shortage in button-board has led many to substitute sheet rock, which comes in sheets 4 feet by 8 feet, in which they have bored holes one-half inch in diameter in such a way that no point on the surface of the plaster-board is more than three inches from the perimeter of the nearest hole. What is the minimum number of holes that can be used?

MATHEMATICAL MISCELLANY

Edited by
MARIAN E. STARK

Let us know (briefly) of unusual and successful programs put on by your Mathematics Club, of new uses of mathematics, of famous problems solved, and so on. Brief letters concerning the MATHEMATICS MAGAZINE or concerning other "matters mathematically" will be welcome. Address: MARIAN E. STARK, Wellesley College, Wellesley 81, Mass.

Any teacher of mathematics in a secondary school or college who has taught *five years or less* is urged to send this department a statement of not more than one short paragraph, containing what he considers the most important advice to give a teacher of our subject as he starts his career. The writer's name should be given, together with the name of the institution where he teaches and the number of years he has taught. The most interesting statements received will appear in this department from time to time.

One report on a hobby of a mathematics teacher after "age 65" is at hand. A Harry Wheeler of Worchester, Mass., formerly of Clark University, has taken up the invention of games, more or less mathematical.

Advice needed! What can be done with a college student who can do A work but is perfectly satisfied with C? Such students are very infrequent, but exceedingly irritating. Of course, they have a right to choose how they will spend their time and energy. But how may one inspire them to a better choice than they are making?

"The Duodecimal Society of America is engaged in educating the public in counting by twelves, instead of tens, and in the application of the 12-Base in mathematics, weights and measures, and other branches of pure and applied science. Its Annual Awards are conferred on those who have made outstanding contributions to the development of duodecimals and to the advancement of the purposes of the Society."

New York. THE DUODECIMAL SOCIETY OF AMERICA.

"The calculation of π to many places of decimals has had a fascination for many computers and various formulae have been used for the purpose. In 1841 William Rutherford, using a formula of Euler (1764).

$$\pi/4=4\tan^{-1}1/5-\tan^{-1}1/70+\tan^{-1}1/99,$$

calculate the value of π correct to 152 places of decimals. Using the formula

$$\pi/4=\tan^{-1}1/2+\tan^{-1}1/5+\tan^{-1}1/8$$

Zaxharias Dase published in 1844 the correct value to 200 places of decimals. In 1853, with the formula of Machin (1706).

$$\pi/4=4\tan^{-1}1/5-\tan^{-1}1/239,$$

Rutherford extended his computation of the value of π correct to 440 places of decimals. In the same year William Shanks, using Machin's formula, published a value to 607 places of decimals which was extended to 707 in a value published in 1873. In 1854 Richter's independent calculation of the value of π showed agreement with the result of Shanks, through the five hundredth decimal

"Using the formula

$$\pi/4=3\tan^{-1}1/4+\tan^{-1}1/20+\tan^{-1}1/1985.$$

Mr. D. F. Ferguson, of the University of Manchester, calculated π to 710 places of decimals and showed (1946) that Shanks' value for π was quite incorrect from the 529th decimal place on. Using the Machin formula Dr. John W. Wrench, Jr., of Washington, D. C., and Mrs. Levi B. Smith of Talbolton, Georgia, recalculated the value of π from the 500th to the 808th place of decimals. This result which is published in *Mathematical Tables and Other Aids to Computation*, for April, 1947, may be regarded as a companion to the value of e, computed to 808 places of decimals by Peder Pedersen, also published in the above-mentioned periodical, April, 1946.

"To Euler (1707-1783) is due an extraordinary formula connecting the 'five most important numbers in mathematics', namely: $e^{i\pi}+1=0$.

Our notations e, $i=\sqrt{-1}$, are also due to Euler. The use of π as the ratio of the circumference of a circle to its diameter was first given by William Jones, in his *Synopsis Palmariorum Matheseos*, London, 1706, p. 243."

Correction. In the September-October number of the MAGAZINE on page 59 read Peder Pedersen for Peder Pederson.

At Duke University an Institute for Teachers of Mathematics was held, August 5-15, 1947, with the general theme "Mathematics at Work". Any reader who attended the Institute is invited to report to this department briefly on interesting results there. One or more reports could be published.

Princeton University has published a pamphlet concerning the Conference on the Problems of Mathematics held in connection with its Bicentennial Celebration in 1946. Very interesting reading! Professor Solomon Lefschetz was the director of the Conference.

A student, desiring to perform the integration

$$\int \frac{d\theta}{\tan\theta\sec^2\theta-\tan\theta}. \tag{1}$$

notes that the cotangent is the reciprocal of the tangent, and hence any cofunction is the reciprocal of the function. He rewrites the integrand as follows:

$$\int \frac{1}{\tan\theta\sec^2\theta-\tan\theta}=\cot\theta\csc^2\theta-\cot\theta. \tag{2}$$

He then performs correctly the revised integration; and produces the correct answer. Irritating fact: (2) is an identity.

Dartmouth College. B. H. BROWN.

Colonel W. E. Byrne of the MAGAZINE's editorial staff is going to report to this department from time to time "news of the present status of mathematicians in France, their needs, etc." We learn from him that M. Gaston Julia has just returned from lecturing in Sweden and Norway. Can any reader supply news of our colleagues in other foreign countries?

Correct Results by an Unorthodox Method. While resolving improper fractions into an integral function and two or more partial

fractions in order to integrate, a student found that after obtaining the integral rational part of the answer he could sometimes find the partial fractions by an unorthodox procedure. This was to set the original fraction

$$\frac{f(A)}{\phi(X)}=\frac{A_1}{X-r_1}+\frac{A_2}{X-r_2}+\cdots+\frac{A_n}{X-r_n}$$

where the denominators on the right are factors of $\phi(X)$. The class found that, although this is not a sensible method to use, it works when the $X-r_1$ are distinct as follows:

$$f(x)=F(x)\ \phi(x)+R(x).$$

For $x=r_i$, $\phi(x)=0$, therefore $f(r_i)=R(r_i)$. The $R(r_i)$ is the number obtained on the left in the usual procedure of substituting r_i for x to obtain $A_1, A_2, \cdots A_n$, Obviously, when ϕ contains repeated factors the method fails as we must substitute numbers other than r_i.

American International College. H. E. BOWIE.

Professor Albert E. Meder, Jr. tells us that the following song was sung to the tune of "The Lorelei" at a party of the Mathematics Club of the New Jersey College for Women, and that it is a translation of a German poem by K. Lasswitz, 1880, which appears on pages 27 and 28 of Leitzmann's *Lustiges und Merkwurdiges von Zahlen und Formen*, Breslau, 1928.

SAD BALLAD OF THE JEALOUS CONES

I. There once were two cones closely plighted
In brotherly love and in bliss
Upon the same axis united
Their vertices joined in a kiss
And then a smooth plane came a flying
(Its smoothness was chief of its charms)
And quickly, without even trying
Seized one of the cones in its arms.

II. An element chose at its pleasure—
Then parallel took up its place
While joy settled down without measure
Upon the parabola's face.
The vertical cone found it tragic
For never the gracious smooth plane
Could cut it—not even by magic—
It suffered in agonized pain.

III. It urged and cajoled without ceasing
And encouraged the plane soon to come
With coaxing and begging and teasing
It promised all comforts of home.
The plane, which enjoyed being flattered
Listened and heeded, entranced;
A hyperbola now only mattered
For beauty would thus be enhanced.

IV. Serenely, with great satisfaction,
The plane intersected each cone,
But these were annoyed by its action,
Each jealously deemed it its own
Each jabbed with its point at its brother
And tore at the unhappy plane
On their axis they turned round each other
And struggled with might and with main.

V. The plane cried, "Alas, cease your railings
  You're turning me round and round
And though I despise finite failings
  I'll become an ellipse I'll be bound."
The cones, tho' their struggle was needless
  Would cease not, not list to its call;
And they pulled at each other, still heedless
  Of how the poor conic grew small.

VI. As the angry cones still madly battled
  A terrible shriek rent the air;
The plane thru the vertex had rattled—
  The section had gone now for fair.
Two surfaces battered and rumpled
  Forever and ever apart
A crushed little point, badly crumpled—
  The tragedy's end breaks my heart!!